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XIII.

THE PHASES OF THE MOON.

By Arthur Searle.

Presented April 9, 1884.

During the passage of the Moon through the ordinary course of its phases, the observed variations of the total amount of light which we receive from it are not readily accounted for by the accepted laws of optics. The *Photometrische Untersuchungen* of Zöllner (Leipzig, 1865) contains the results of previous inquiries into this subject, including those of Zöllner himself. Since then, so far as I have learned, no important observations or theories relating to the phases of the Moon have been made public. Additional knowledge of these phases would have some useful applications to other subjects, and in particular to that theory of the zodiacal light which ascribes it to sunlight reflected from meteoric matter. From this point of view, an examination of Zöllner's conclusions in the work above named, and an attempt to derive some additional inferences from the material collected by him, may be desirable.

The formula most commonly assumed as an expression of the variation of the total observed light of a distant object, in consequence of its changes of phase, is due to Lambert. The hypothesis on which it depends is that the light received from equally bright surfaces will be proportional to their apparent areas. In other words, Lambert asserts the quantity of light received from a small distant object to be proportional, other things being equal, to the magnitude of its orthographic projection upon a plane perpendicular to the line of sight. The validity of the assertion was rather feebly supported by Lambert himself, and has never been clearly established. Zöllner, however, has pointed out some considerations favorable to Lambert's hypothesis; and it will probably continue to find employment, in the want of a more complete investigation of the question to which it relates.

In addition to this special hypothesis, Lambert accepts the ordinary photometric rule, (which appears to be almost a necessary consequence of the rectilinear transmission of light,) that the actual illumination of a small surface, at a great distance from the source of light, will be proportional to the magnitude of its orthographic projection upon a plane perpendicular to the incident rays; that is, if the surface is plane, to the cosine of the angle of incidence. Its apparent illumination, then, upon Lambert's theory, will be proportional, at any single point, to the product of the cosines of the angles of incidence and emanation at that point, if we understand the angle of emanation to signify the angle between the normal to the illuminated surface and its direction from the observer. This theory assumes that the reflection of light from the given surface takes place irregularly, specular reflection being altogether neglected. For this reason, as well as for others stated by Zöllner (p. 24), Lambert's formula seems likely to be fully applicable only to somewhat translucent objects. A strictly opaque object, if smooth, will probably exhibit the phenomena of specular reflection to such an extent as to destroy the value of the formula; if rough, the theoretical quantity of irregularly reflected light will be variously reduced in different phases by the shadows of the prominences.

With the assumptions above explained, Lambert arrived at the result that the light received by irregular reflection from a distant sphere would vary with the phases of the sphere in accordance with the expression $\sin v - v \cos v$, where v denotes the angular magnitude of the phase. It may also be defined as the exterior angle, at the illuminated object, of the triangle formed by the straight lines connecting that object, the source of light, and the observer. If the total quantity of light received from the sphere when in exact opposition is regarded as unity, the quantity in any other phase will accordingly be denoted by $\frac{1}{\pi}(\sin v - v \cos v)$, provided that we regard as constant quantities the intensity and magnitude of the source of light, the magnitude and reflecting power of the sphere which it illuminates, and the distance of the sphere both from the source of light and from the observer. The coefficients depending upon these quantities will accordingly be neglected in the present inquiry.

Zöllner, in order to simplify the discussion of the lunar phases, shows that the variable factor $\sin v - v \cos v$ will be applicable, on Lambert's hypothesis, to the phases of a right cylinder, with its axis perpendicular to the plane of the triangle having its vertices at the source of light, the cylinder, and the observer. The proposition may apparently be extended to any surface of revolution subjected to the same condition. Let dl denote an element of the generating line, dh its orthographic projection on the axis, and dp its orthographic projection on a fixed plane containing the axis. Let r denote the perpendicular let fall upon the axis from dl, and θ the angle made by r with a perpendicular to the fixed plane. The arc described by a point in dl during the small fraction of a revolution which may be denoted by $d\theta$ is expressed by $rd\theta$; the projection of this arc upon the fixed plane is $r \cos \theta \, d\theta$. The projection of the surface, having dl and $rd\theta$ for its altitude and base, has $r \cos \theta \, d\theta$ for its base, but dh, not dp, for its altitude. Accordingly, if l denotes the entire length of the generating line, and h its projection on the axis, the ratio of the element, described by the generating line during its movement through $d\theta$, to its projection on the fixed plane, is expressed by $\frac{l}{h \cos \theta}$. Hence, for different values of θ, the projections of the corresponding elements of the surface are proportional to $\cos \theta$. If the projection of each of the surfaces described by the elements of l during its movement through $d\theta$ is substituted for that surface, and again orthographically projected upon a new plane containing the axis, the final projections of corresponding elements of the surface will appear by the same course of reasoning to be proportional to $\cos \theta \cos \gamma$, where γ denotes the angle made by r with the perpendicular to the new plane. We may regard θ as the angle of incidence of parallel rays of light upon a cylinder circumscribing the supposed surface, and having the same axis; γ may be regarded as the corresponding angle of emanation. On Lambert's hypothesis, the ratio of light received from the given surface in different phases will be the ratio of quantities obtained by the integration of $\cos \theta \cos \gamma \, d\theta$ between limits determined for each phase by the extent of the illuminated surface. Denoting the magnitude of the phase, as above, by v, and regarding θ and γ as positive when they are measured from the normal towards the terminator, the limiting values of θ are $-\frac{1}{2}\pi$ at the terminator, and $-(\frac{1}{2}\pi - v)$ at the illuminated limb; those of γ are $\frac{1}{2}\pi - v$ at the terminator, and $\frac{1}{2}\pi$ at the illuminated limb. For any value of γ, the corresponding value of θ is $\gamma - (\pi - v)$. Hence $\cos \theta \cos \gamma \, d\theta = - \cos \gamma \cos (v + \gamma) \, d\gamma$, the indefinite integral of which is $-\frac{1}{2} [\sin \gamma \cos (v + \gamma) + \gamma \cos v]$. Integrating between the given limits of γ, we have the result $\frac{1}{2} (\sin v - v \cos v)$; the constant factor $\frac{1}{2}$ indicates limitation to half the surface, and vanishes from the ratio between the quantities of light received from different phases. The integration may be somewhat simplified by employing the mean of θ and γ as the variable, with the aid of the general formula

$$\sin (a + b) \sin (a - b) = \tfrac{1}{2} (\cos 2b - \cos 2a).$$

To find the ratio of light, for a given phase, between two different surfaces of revolution, we have only to determine the corresponding ratio of light for single elements of each surface, with the same values of θ and of γ for each element; the values to be assumed in practice are $\theta = \gamma = 0$. In the case of the sphere and the circumscribing cylinder, the height of which is equal to the diameter of the sphere, let ζ denote the angle between the axis and the radius drawn to any point in that element of the sphere for which θ and γ vanish. At this point, the angles of incidence and emanation are each equal to the complement of ζ; and the quantity of light received from the immediate vicinity of the point will have the ratio $\sin^2 \zeta$ to that received from an equal surface at any point of the corresponding element of the cylinder. But as the width of the spherical element, at the given point, is the product by $\sin \zeta$ of the width of the cylindrical element, the quantity of light received from the element of the sphere, at the given point, will have the ratio $\sin^3 \zeta$ to that received from an equal length of the element of the cylinder. Accordingly, if the radius of the sphere is the unit of length, the ratio of the quantities of light received from the spherical and cylindrical elements is $\frac{1}{2}\int_0^\pi \sin^3 \zeta \, d\zeta$. Changing the variable to $-\cos \zeta$, which we may represent by x, we have

$$\tfrac{1}{2}\int_0^\pi \sin^3 \zeta \, d\zeta = \tfrac{1}{2}\int_{-1}^{+1} (1 - x^2)\, dx = \tfrac{2}{3},$$

the ratio found by Zöllner (p. 40), from an independent determination of the phases of the cylinder, combined with Lambert's similar proposition relating to the sphere. It is also sufficiently apparent, although not distinctly set forth, in Seidel's statement of Lambert's demonstration, that this ratio exists.

If, instead of the sphere, we consider a right cone, inscribed in a cylinder of the same height and base, the coefficient required to reduce the phases of the cylinder to those of the cone is readily perceived to be $\frac{1}{2} \cos \chi$, in which χ denotes the inclination of the generating straight line to the axis. The direct determination of the phases of the cone is nearly as simple. The cosines of the angles of incidence and emanation will be respectively expressed by $\cos \chi \cos \theta$ and $\cos \chi \cos \gamma$; the slant height of the cone is proportional to $\sec \chi$; and the expression to be integrated becomes

$$-\tfrac{1}{2} \cos \chi \cos \gamma \cos (v + \gamma)\, d\gamma.$$

For a frustum the height of which is insignificant in comparison with the total height of the cone, the coefficient $\frac{1}{2}$ disappears.

Zöllner regarded the analogy existing between the phases of a smooth cylinder and a smooth sphere as an indication that the effect of roughening the surfaces would be somewhat similar in the two cases (p. 51). Although it is doubtful if this extension of the analogy can be carried far, it is still reasonable to think that it may be of use in obtaining some notion of the kind of modification which Lambert's formula will require when it is applied to the phases of a mountainous body like the Moon. Zöllner accordingly considered the consequences of covering the surfaces of the cylinder with furrows parallel to the axis, and obtained a general expression for the phases of this furrowed cylinder without regarding the dimensions of the furrows as insignificant with respect to the diameter of the cylinder. By so regarding them, the expression was afterwards reduced to

$$\sin (v - \beta) - (v - \beta) \cos (v - \beta),$$

if we omit, as before, various factors independent of the phase. In this expression, β signifies the angle between the base and the slope of each ridge, called by Zöllner the angle of elevation. He found his own photometric observations between half and full moon well satisfied by this formula when the value of β was assumed to be 52°.

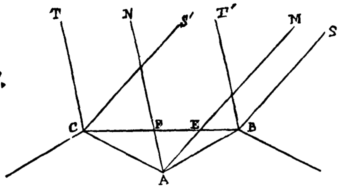
The geometrical significance of the expression

$$\sin (v - \beta) - (v - \beta) \cos (v - \beta)$$

was partially developed by Zöllner (p. 66). A sufficient extension of the process would have enabled him to correct an error in his analysis which, as it happened, did not attract his attention. The expressions for the elementary surfaces F_1 and F_2 (pp. 55 and 56) are not applicable throughout a sufficiently large phase. Both the general expression and the simplified formula above mentioned are therefore incomplete. The correction of the general expression is unnecessary for our present purpose. We may accordingly regard a section of the cylinder perpendicular to its axis as a regularly serrated circle, each serration being so small that the arc forming its base must be considered as a straight line.

In the following figure, let A represent the junction of two adjacent serrations, the vertices of which are at B and C. From B and C draw the parallels BS, CS′, directed towards the source of light, and BT′, CT, directed towards the observer. From A draw AM parallel

Fig. 1.



to BS, and AN parallel to CT, respectively intersecting the straight line BC at E and F.

Upon Lambert's hypothesis, equal amounts of light will emanate from all straight lines at a sufficient distance from the observer, intercepted between the parallels AN, BT′, and equally illuminated; that is, so illuminated that equal lengths of different lines receive equal quantities of incident light. This condition will be fulfilled (if the intensity of the illumination remains the same for all the lines) when the lines make equal angles with the rays of light. Suppose the furrowed cylinder to be inscribed in a smooth cylinder, and BC to represent a sensibly straight portion of the circle bounding the intersection of the new cylinder with the plane containing the lines AB, AC. At the same time, let the system of lines directed to the source of light be revolved about B as a centre through the angle ABC, so that the inclination of BS to BC may become equal to the former inclination of BS to BA. The illumination of the line BC, or of any part of it, as BF, will now be equal to the former illumination of BA. But BF and BA are both intercepted between the parallels AN, BT′; hence the quantity of light emanating from BF is equal to that formerly emanating from BA. The revolution of BS about B has increased the angle SBT′ by the amount ABC; but SBT′ is the supplement of the magnitude of the phase, and ABC is equal to the angle of elevation of the serrations. Accordingly, when the phase of the smooth cylinder is less than the phase of the furrowed cylinder by an amount equal to this angle of elevation, equal quantities of light will reach the observer from the surfaces of which BF and BA are respectively the sections.

If the position of the observer is changed, so that the angle CBT′ becomes equal to the angle of elevation ABC or ACB, the point F coincides with C; in this case, equal quantities of light are received from BC in the smaller, and from BA in the greater phase. For still smaller values of CBT′, we have merely to substitute for BA that part of it still visible to the observer. Hence, while the phase remains so small that no pair of adjacent slopes, like AB and AC, are at once illuminated and visible, the light received from the visible parts of slopes facing towards the terminator is equal to that received from the continuous surface of the same region of the smooth cylinder when the phase is diminished by an amount equal to the angle of elevation. In this case, the formula obtained by Zöllner is partially applicable, since it is accordant with the fact that no light will be received from the furrowed cylinder until the phase is larger, by an amount equal to the angle of elevation, than the initial phase of the smooth cylinder. As this was the only point considered in Zöllner's geometrical interpretation of his formula, the result of his analysis appeared to be confirmed.

In considering the quantities of light received from the slopes facing the illuminated limb, it will be convenient to assume what is approximately true in the case of the Moon, that during the changes of phase the angles of emanation at given points of the illuminated surface are constant. Upon Lambert's hypothesis, equal quantities of light will emanate from lines equally inclined to the line of sight, but of different lengths, provided that these lines have received equal quantities of incident light. In the figure, the lines CA, CE, receive equal quantities of incident light, and if we now suppose the phase diminished by the revolution of CT, and the lines parallel to it, about C as a centre, through the angle ACB, the observed light from CE will be equal to the light formerly received from CA. Since, however, we have assumed the direction of CT with respect to CA to be invariable, this result signifies that, when the phase is diminished, a quantity of light equal to that formerly received from CA is now received, not from CE, but from an equal arc of the circumference of which CE is part, situated as far from the new terminator as CE was from the original terminator. It may also be observed that $BF + CE > BC$. Hence, in order to assume, with Zöllner, that the light received from the smooth cylinder in the reduced phase is equal to that previously received from the furrowed cylinder, we must assume at the same time an increase of reflecting power in part of the smooth cylinder. Leaving other geometrical considerations to suggest themselves, we may now construct the formula required to represent the phases of the furrowed cylinder.

Returning to the figure, let s denote the length of the slope AB or AC; β, the angle of elevation ABC or ACB; $\pi - v$, the supplement SBT′ of the magnitude of the phase; θ, the angle of incidence on BC, of which MEB is the complement; and γ, the corresponding angle of emanation, the complement of which is NFC. As before, θ and γ will be regarded as positive when measured from the normal to BC towards the terminator, and the constant factors expressing the intensity of the light and the dimensions of the cylinder will be neglected. The short arc BC may be represented by $d\gamma$, if we consider the vertex of an angle equal to γ as placed in the axis of the cylinder. The quantities of light received from the slopes AB and AC will be respectively expressed by $s \cos(\theta - \beta) \cos(\gamma - \beta)$ and $s \cos(\theta + \beta) \cos(\gamma + \beta)$. The value of s is $\frac{1}{2} \sec \beta \, d\gamma$, and that of θ is $\gamma - (\pi - v)$.

These expressions do not represent the consequences which result from the opacity of the surfaces considered, so that the integration must be confined to the limits within which the whole surface of every slope is illuminated and visible. The slopes facing towards the terminator for which $\theta - \beta < -\frac{1}{2}\pi$ will not be illuminated, and the slopes respectively opposite them will be partly in the shade. The slopes facing towards the illuminated limb for which $\gamma + \beta > \frac{1}{2}\pi$ will not be visible, and those opposite to them will be partly hidden. The required integral is therefore

$$-\tfrac{1}{2} \sec \beta \left[\int_{\frac{1}{2}\pi - v}^{\frac{1}{2}\pi - 2\beta} \cos(\gamma - \beta) \cos[v + (\gamma - \beta)] \, d(\gamma - \beta) \right.$$
$$\left. + \int_{\frac{1}{2}\pi - v + 2\beta}^{\frac{1}{2}\pi} \cos(\gamma + \beta) \cos[v + (\gamma + \beta)] \, d(\gamma + \beta) \right]$$

the terms of which have the same form as that of an expression already found (p. 312). If $v = 2\beta$, as well as when $v < 2\beta$, no finite region exhibits slopes wholly visible and illuminated, so that this formula is then inapplicable. For those cases in which $v > 2\beta$, the sum of its two equal terms becomes, on reduction,

$$\tfrac{1}{2} \sec \beta \, [\cos 2\beta \sin(v - 2\beta) - (v - 2\beta) \cos v].$$

When $\beta = 0$, the limits of the integration coincide with those of the phase, and the result becomes $\frac{1}{2}(\sin v - v \cos v)$, as before. When $\beta = \frac{1}{2}\pi$, v cannot exceed 2β.

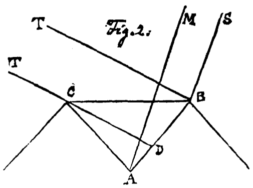


Fig. 2.

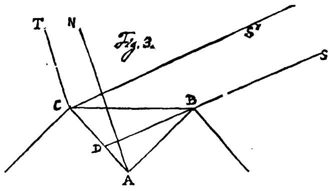


Fig. 3.

To complete the solution, we have still to consider the partly visible and the partly illuminated slopes. The first case is exhibited in Fig. 2, and the second in Fig. 3. The length of BD, the visible portion of the slope AB, is readily found to be $\frac{\cos\gamma}{\cos(\gamma-\beta)}\,d\gamma$; that of CD, the illuminated portion of AC, is $\frac{\cos\theta}{\cos(\theta+\beta)}\,d\gamma$. The expressions to be integrated in the two cases are respectively $\cos\gamma\cos(\theta-\beta)\,d\gamma$, and $\cos\theta\cos(\gamma+\beta)\,d\gamma$. The first may be reduced to

$$-\cos\gamma\cos[(v-\beta)+\gamma]\,d\gamma;$$

the second to

$$-\cos(v+\gamma)\cos(\gamma+\beta)\,d\gamma,$$

and this, again, to

$$-\cos(\gamma+\beta)\cos[(v-\beta)+(\gamma+\beta)]\,d(\gamma+\beta).$$

To determine the limits of integration, we are to consider that while $v<\beta$ no illuminated portion of the furrowed cylinder can be visible; and also that for values of v between β and 2β two illuminated portions of the cylinder will be visible, separated by a dark interval. Part of each slope facing towards the terminator will be illuminated, and will include a visible portion, for values of γ extending from $\frac{1}{2}\pi$ to $\frac{1}{2}\pi-(v-\beta)$. These are accordingly the limiting values of γ in the integration of $-\cos\gamma\cos[(v-\beta)+\gamma]\,d\gamma$. Part of each slope facing towards the illuminated limb will be visible, and will include an illuminated portion, for values of γ extending from $\frac{1}{2}\pi-\beta$ to $\frac{1}{2}\pi-v$, where the terminator is reached. The expression

$$-\cos(\gamma+\beta)\cos[(v-\beta)+(\gamma+\beta)]\,d(\gamma+\beta)$$

is consequently to be integrated between the limits

$$\tfrac{1}{2}\pi \quad\text{and}\quad \tfrac{1}{2}\pi-(v-\beta).$$

It is only necessary to compare these expressions, and their limits, with the results obtained above (p. 312), to see that each of the definite integrals required will have the value

$$\tfrac{1}{2}[\sin(v-\beta)-(v-\beta)\cos(v-\beta)];$$

it is also apparent from geometrical considerations that this should be the result for values of v between β and 2β. When $v>2\beta$, the inferior limit of the integration of $-\cos\gamma\cos[(v-\beta)+\gamma]\,d\gamma$ remains constant at $\frac{1}{2}\pi-\beta$; and the superior limit of the other integration becomes $\frac{1}{2}\pi-(v-2\beta)$. The result of each integration is now

$$\tfrac{1}{2}[\sin\beta\cos(v-2\beta)-\beta\cos(v-\beta)].$$

When $v = 2\,\beta$, this result coincides, as it should, with that from

$$\tfrac{1}{2}\,[\sin(v - \beta) - (v - \beta)\cos(v - \beta)].$$

The collection of the separate results of the inquiry furnishes the following rule for computing, upon Lambert's hypothesis, the phases of a right cylinder, with a surface occupied by regular furrows, parallel to the axis, and of dimensions too small to be comparable to those of the cylinder, which is supposed to be placed at a great and constant distance from the source of light and also from the observer, with its axis perpendicular to the rays of incident light and to the line of sight: —

If we adopt as the unit of measurement the quantity of light received from a smooth cylinder equal in height and diameter to the furrowed cylinder, when the magnitude of the phase is an entire semi-circumference, denoting this magnitude, for any phase, by v; the angle of elevation, formed by the base and slope of each ridge, by β; and the observed quantity of light, by L; then, if $\beta = 0$,

$$L = \frac{1}{\pi}\,(\sin v - v\cos v). \qquad (1)$$

If $\beta > 0$, and $v < \beta$, $L = 0$.

If $v > \beta > 0$, and $v < 2\,\beta$,

$$L = \frac{2}{\pi}\,[\sin(v - \beta) - (v - \beta)\cos(v - \beta)]. \qquad (2)$$

If $v = 2\,\beta$,

$$L = \frac{2}{\pi}\,(\sin\beta - \beta\cos\beta). \qquad (3)$$

If $v > 2\,\beta > 0$,

$$L = \frac{2}{\pi}\,[\sin\beta\cos(v - 2\,\beta) - \beta\cos(v - \beta)]$$
$$+ \frac{1}{\pi}\sec\beta\,[\cos 2\,\beta\sin(v - 2\,\beta) - (v - 2\,\beta)\cos v]. \qquad (4)$$

By the collection of the terms of equation (4) which contain respectively $\sin v$ and $\cos v$, the equation is reduced to

$$L = \frac{1}{\pi}\sec\beta\,[(1 - \beta\sin 2\,\beta)\sin v - (v - 2\,\beta\sin^2\beta)\cos v]. \qquad (5)$$

Equation (5) is readily transformed to

$$L = \frac{1}{\pi}\sec\beta\,[\sin v - v\cos v - 2\,\beta\sin\beta\sin(v - \beta)]. \qquad (6)$$

Since the expression $\log\frac{2}{3\,\pi}\,(\sin v - v\cos v)$ has been tabulated by Seidel,* equation (6) is somewhat better adapted to numerical computation than equation (5). Equation (2) may be very readily computed from Seidel's table.

By means of equations (2) and (6), numerical expressions may be obtained for the quantities of light corresponding to the phases of the furrowed cylinder for assumed values of β, in terms of the quantity of light received from the smooth cylinder in opposition. If we adopt as the unit the quantity of light received from the furrowed cylinder itself when $v = \pi$, we must divide each result by the corresponding value of the expression $\frac{1}{\pi} \sec \beta \, (\pi - 2 \, \beta \sin^2 \beta)$.

The differential coefficient of the value of L in equation (6), taken with respect to v, is

$$\frac{1}{\pi} \sec \beta \, [v \sin v - 2 \, \beta \sin \beta \cos (v - \beta)].$$

The value of L, accordingly, if $\beta > 0$, does not reach a maximum when $v = \pi$. For this value of v, L is greatest when the value of β is about 38°; when $\beta = 60°$, $L = 1$; and for the limiting value $\beta = 90°$, $L = \frac{2}{\pi}$. This last result appears from equation (3); equation (6) is not applicable to the case without previous reduction to (3), since v cannot exceed $2 \, \beta$ when $\beta = \frac{1}{2} \pi$.

Numerical values of L for six assumed values of β are given in Table I. The unit of light, in this part of the table, is the quantity received from the smooth cylinder when $v = \pi$. The last three columns contain results derived from the work of Zöllner. The first of these columns contains the values of L according to his formula, when $\beta = 52°$, and when the amount of light received upon this supposition is regarded as unity if $v = \pi$. The next two columns relate to the twenty-two sets of observations, the results of which are given on page 102 of his work. These results are here arranged in accordance with the values of v, for which, the phase being always large, Zöllner substitutes the corresponding elongations of the Moon. The values of v are here printed in Italics when the observations were made after full moon. The observation made when $v = 179°$ is assumed by Zöllner to be in accordance with his formula, in order to compare the other observations with the results of the theory. This assumption is here retained.

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* Untersuchungen über die Lichtstärke der Planeten Venus, Mars, Jupiter und Saturn. München, 1859. (Pp. 100–102 contain the table above mentioned.)

TABLE I.

v	Theoretical Results for Cylinder.						Zöllner's Formula.	Zöllner's Observations.	
	$\beta = 0°$	$\beta = 30°$	$\beta = 38°$	$\beta = 45°$	$\beta = 60°$	$\beta = 90°$	$\beta = 52°$	v	Ratio.
5°	0.000	.....	.....	.....	.....	.....	.....	.....	.....
10	0.001	.....	.....	.....	.....	.....	.....	.....	.....
15	0.002	.....	.....	.....	.....	.....	.....	.....	.....
20	0.004	.....	.....	.....	.....	.....	.....	.....	.....
25	0.009	.....	.....	.....	.....	.....	.....	.....	.....
30	0.015	.....	.....	.....	.....	.....	.....	.....	.....
35	0.023	0.000	.....	.....	.....	.....	.....	.....	.....
40	0.034	0.001	0.000	.....	.....	.....	.....	.....	.....
45	0.048	0.004	0.000	.....	.....	.....	.....	.....	.....
50	0.065	0.009	0.002	0.000	.....	.....	.....	.....	.....
55	0.085	0.017	0.005	0.001	.....	.....	0.000	.....	.....
60	0.109	0.030	0.012	0.004	.....	.....	0.000	.....	.....
65	0.136	0.047	0.022	0.009	0.000	.....	0.002	.....	.....
70	0.166	0.068	0.036	0.017	0.001	.....	0.005	.....	.....
75	0.200	0.094	0.055	0.030	0.004	.....	0.010	110	0.203
80	0.236	0.125	0.079	0.047	0.009	.....	0.018	*111*	0.146
85	0.276	0.161	0.109	0.069	0.017	.....	0.028	*118*	0.204
90	0.318	0.201	0.144	0.097	0.030	.....	0.043	122	0.271
95	0.363	0.245	0.186	0.130	0.047	0.000	0.062	*128*	0.291
100	0.410	0.293	0.228	0.170	0.069	0.001	0.084	134	0.361
105	0.458	0.343	0.278	0.215	0.097	0.004	0.112	*138*	0.381
110	0.508	0.397	0.330	0.265	0.131	0.009	0.144	139	0.439
115	0.558	0.453	0.387	0.320	0.171	0.017	0.181	140	0.471
120	0.609	0.511	0.447	0.378	0.218	0.030	0.223	*141*	0.417
125	0.659	0.569	0.507	0.440	0.271	0.047	0.270	147	0.486
130	0.708	0.628	0.569	0.503	0.331	0.069	0.321	*152*	0.570
135	0.755	0.686	0.631	0.568	0.395	0.097	0.377	152	0.562
140	0.800	0.743	0.693	0.634	0.464	0.131	0.437	153	0.635
145	0.842	0.798	0.754	0.699	0.535	0.171	0.501	*153*	0.579
150	0.881	0.850	0.812	0.763	0.607	0.218	0.568	156	0.714
155	0.915	0.902	0.867	0.824	0.680	0.272	0.637	161	0.684
160	0.944	0.943	0.918	0.882	0.751	0.332	0.709	*167*	0.826
165	0.968	0.981	0.965	0.936	0.820	0.399	0.782	169	0.888
170	0.985	1.014	1.005	0.984	0.886	0.473	0.855	172	0.922
175	0.996	1.040	1.039	1.026	0.946	0.552	0.928	*175*	0.872
180	1.000	1.058	1.066	1.061	1.000	0.637	1.000	179	0.986

It is apparent upon inspection of the table, and will become still more obvious upon a graphical arrangement of the data, that β must be as large as 60° in order to represent Zöllner's observations of the Moon by means of our equation (6). For this value of β we have $L = 1$ when $v = \pi$, so that the comparison can be made at once in Table I. without any change of scale. A somewhat greater value of β would perhaps better represent the observations, but this point does not seem to require close examination. In any case, the changed form of the curve representing the successive values of L, as v increases, renders it less accordant than Zöllner's own formula with the results of his observations. Upon examining the residuals, I find the results to be as follows: —

	Sum of Positive Residuals.	Sum of Negative Residuals.	Average Deviation.
By Zöllner's formula,	0.326	0.314	0.029
By equations (2) and (6), when $v = 60°$,	0.166	0.648	0.037
By assuming L to increase 0.137 for an increase of 10° in v,	0.518	0.431	0.043

It appears, then, that Zöllner's formula probably represents the actual phases of the Moon, near opposition, better than it represents the phases of a furrowed cylinder, computed upon Lambert's hypothesis. The similarity between the two objects is certainly not great enough to make it at all surprising that their phases should differ.

It may now be asked whether any useful inference can be derived from the theoretical investigation undertaken by Zöllner, and re-examined in the preceding pages. Before deciding this question in the negative, it may be well to review the facts which have hitherto come separately to our notice.

In the first place, while Lambert's hypothesis of the emanation of light stands greatly in need of experimental verification, it is still too accordant with some familiar facts to be regarded as a mere conjecture. Smooth rounded surfaces under a diffused illumination (as, for example, drifts of snow under a cloudy sky) have a general appearance of uniform brightness, which makes it difficult to distinguish their outlines against a background of the same color. It may well be that the true law of the emanation of light is not so simple as Lambert regarded it, while yet his hypothesis may furnish a tolerable approximation to the truth. The corresponding hypothesis with regard to incident light is so universally accepted, that it needs no verbal support, although additional experimental evidence with regard to it, and especially with regard to the modifications which it may need in practice, is highly desirable.

Accepting these two hypotheses, Lambert's formula for the phases of a smooth distant sphere follows as a matter of course; and the attempt has been made above to show that any smooth surface of revolution, with its axis perpendicular to the plane determined by the positions of the source of light, the illuminated body, and the observer, will resemble the sphere in its phases. It is therefore somewhat surprising to find that the actual phases of the Moon, at least near opposition, disagree so decidedly with the results of Lambert's formula.

Herschel's observations, as collected and reduced by Zöllner, exhibit a still greater divergence from Lambert's formula than those of Zöllner himself. Bond also obtained a like result from his own photometric observations.

It seems, accordingly, to be well worth while to investigate the results of Lambert's hypothesis when applied to a rough body of some kind. As a first step in this direction, Zöllner's discussion of the phases of a roughened cylinder was apparently judicious, although he seems to have laid too much stress on the analogy between a smooth cylinder and a smooth sphere. But the accidental error in his analysis, which has been noticed above, would have made it difficult to decide, in the absence of further examination, whether any confidence could be placed in his conclusion. The formula obtained in the preceding pages, however, agrees with Zöllner's in its most important peculiarity. We find that the phases of a furrowed cylinder, like the observed phases of the Moon, must exhibit a comparatively rapid increase of light towards opposition, and that this increase ends abruptly when opposition is reached, beginning to diminish abruptly as soon as opposition is passed. The great angle of elevation required in the ridges of the cylinder, before its computed phases begin to resemble those of the Moon, makes it probable that other forms of roughness besides that exhibited by the furrowed cylinder are concerned in the problem. But the degree of success attending the investigation proposed by Zöllner is sufficient to encourage further examination. Undoubtedly, however, additional observations are much more important, if not more interesting, than these theoretical reconciliations of Lambert's hypothesis with the facts of nature.

The meteoric theory of the zodiacal light is affected by inquiries into the phases of rough bodies, since it is highly probable that the meteors assumed to produce the light are of irregular shape and of uneven surface. An important result already obtained is, that minute but abrupt irregularities may be expected materially to modify the phases of the bodies which exhibit them, and that, upon any plausible hypothesis of reflection, such irregularities will tend to produce a relatively great amount of light in the direction opposite to that of the Sun. A few obvious remarks naturally present themselves with regard to the probable effects to be expected from different kinds of irregularity in surfaces; but these effects need some mathematical examination before anything can be said of them sufficiently definite to be useful.